Math 4650

Topic 6 - Uniform Continuity

Def: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$. We say that f is uniformly continuous on D if for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x, y \in D$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Note: In normal continuity δ depends on both ε and a point $x \in D$.
For uniform continuity δ only depends on ε and works at any points in D.

Ex: Let $f(x) = x^2$. Then f is uniformly continuous on $D = [0, 8]$.

pf: Let $\varepsilon > 0$.
Suppose $x, y \in [0, 8]$.
Then

$$
\begin{aligned}
|f(x) - f(y)| = |x^2 - y^2| &= |x+y||x-y| \\
&\le (|x| + |y|)|x-y| \\
&\le (8+8)|x-y| = 16|x-y|.
\end{aligned}
$$

Let $\delta = \frac{\varepsilon}{16}$.
Then if $x, y \in [0, 8]$ and $|x-y| < \delta$ we get

$$|f(x) - f(y)| \le 16|x-y| < 16 \cdot \delta = 16 \cdot \frac{\varepsilon}{16} = \varepsilon.$$

$\blacksquare$

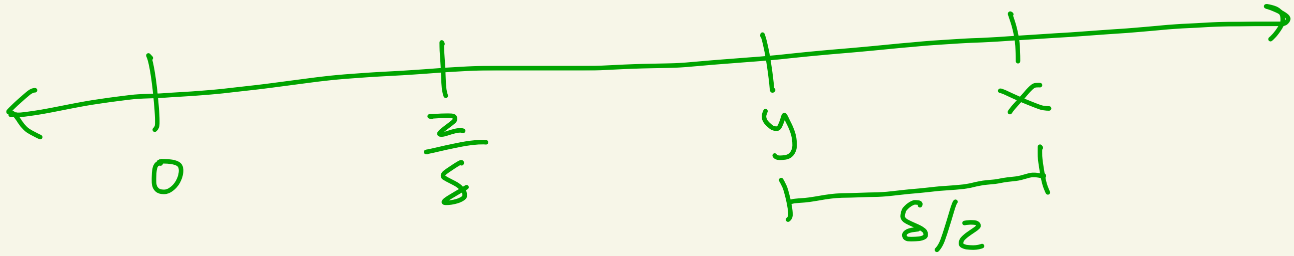
Ex: However, $f(x)=x^2$ is not uniformly continuous on $D=[0,\infty)$, even though it is continuous on D

pf: Let $\varepsilon = 1$.

We will show that for any $\delta > 0$ there exist $x, y \in D$ with $|x-y| < \delta$ but $|x^2-y^2| > \underbrace{1}_{\varepsilon}$

Suppose $\delta > 0$.

Pick any $x, y \in D$ with $|x-y| = \frac{\delta}{2}$ and $x, y > \frac{2}{\delta} > 0$



Then,

$$|x^2-y^2| = |x-y|\,|x+y| \underset{\boxed{x+y>0}}{=} \left(\frac{\delta}{2}\right)(x+y) > \left(\frac{\delta}{2}\right)\left(\frac{2}{\delta}+\frac{2}{\delta}\right)$$

$$= 2 > \underbrace{1}_{\varepsilon}.$$

$\blacksquare$

Theorem: Let $D \subseteq \mathbb{R}$. If $f: D \to \mathbb{R}$ is uniformly continuous on D, then f is continuous on D.

Proof: Let $a \in D$ and $\varepsilon > 0$.

Since f is uniformly continuous on D, there exists $\delta > 0$ where if $x, y \in D$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Thus, if $x \in D$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

So, f is continuous at a. $\blacksquare$

Theorem: If f is continuous on $[a,b]$, then f is uniformly continuous on $[a,b]$

proof:

Suppose f is continuous on $[a,b]$ but not uniformly continuous on $[a,b]$. We will show this leads to a contradiction.

Since f is not uniformly continuous there exists an $\varepsilon_0 > 0$ where for any $\delta > 0$ there exist $x, y \in [a,b]$ with $|x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$.

Thus, for each $n \geq 1$, setting $\delta_n = \frac{1}{n}$, there exist $s_n, t_n \in [a,b]$ with $|s_n - t_n| < \frac{1}{n}$ but $|f(s_n) - f(t_n)| \geq \varepsilon_0$.

So we get two sequences $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ contained in $[a,b]$. By Bolzano-Weierstrass there exist convergent subsequences (s_{n_k}) and (t_{n_k}). Since $a \leq s_{n_k} \leq b$ and $a \leq t_{n_k} \leq b$ we know that if $s = \lim_{k \to \infty} s_{n_k}$ and $t = \lim_{k \to \infty} t_{n_k}$ then $a \leq s \leq b$ and $a \leq t \leq b$ (by HW 2).

Let $\alpha > 0$.

Since $s_{n_k} \to s$, $t_{n_k} \to t$, and $|s_{n_k} - t_{n_k}| < \frac{1}{n_k}$

there must exist $k_0 \in \mathbb{N}$ where

$|s_{n_{k_0}} - s| < \frac{\alpha}{3}$ and $|t_{n_{k_0}} - t| < \frac{\alpha}{3}$

and $|s_{n_{k_0}} - t_{n_{k_0}}| < \frac{\alpha}{3}$.

Thus,

$$|s-t| = |s - s_{n_{k_0}} + s_{n_{k_0}} - t_{n_{k_0}} + t_{n_{k_0}} - t|$$
$$\leq |s - s_{n_{k_0}}| + |s_{n_{k_0}} - t_{n_{k_0}}| + |t_{n_{k_0}} - t|$$
$$< \frac{\alpha}{3} + \frac{\alpha}{3} + \frac{\alpha}{3} = \alpha.$$

So, $|s-t| < \alpha$ for any $\alpha > 0$.

So, $|s-t| = 0$.

Thus, $s - t = 0$ which gives $s = t$.

Hence $f(s) = f(t)$.

Since f is continuous at s and t we have that

$$\lim_{k\to\infty} [f(s_{n_k}) - f(t_{n_k})] = f(s) - f(t) = 0.$$

This implies that there exists n_{k_1} where

$$|f(s_{n_{k_l}}) - f(t_{n_{k_l}})| < \varepsilon_0 .$$

Contradiction! $\blacksquare$

Application (Approximating continuous functions)

Let f be continuous on $[a,b]$.

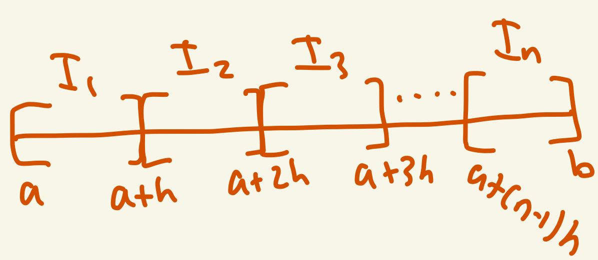
Let $\varepsilon > 0$. Then the following continuous piecewise linear function g_ε defined on $[a,b]$ will satisfy $|g_\varepsilon(x) - f(x)| < \varepsilon$ for all x.

How to define g_ε: Since f is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$. Thus there exists $\delta > 0$ where if $x, y \in [a,b]$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$.

Let n be large enough so that $h = \frac{b-a}{n}$ satisfies $h < \delta$.

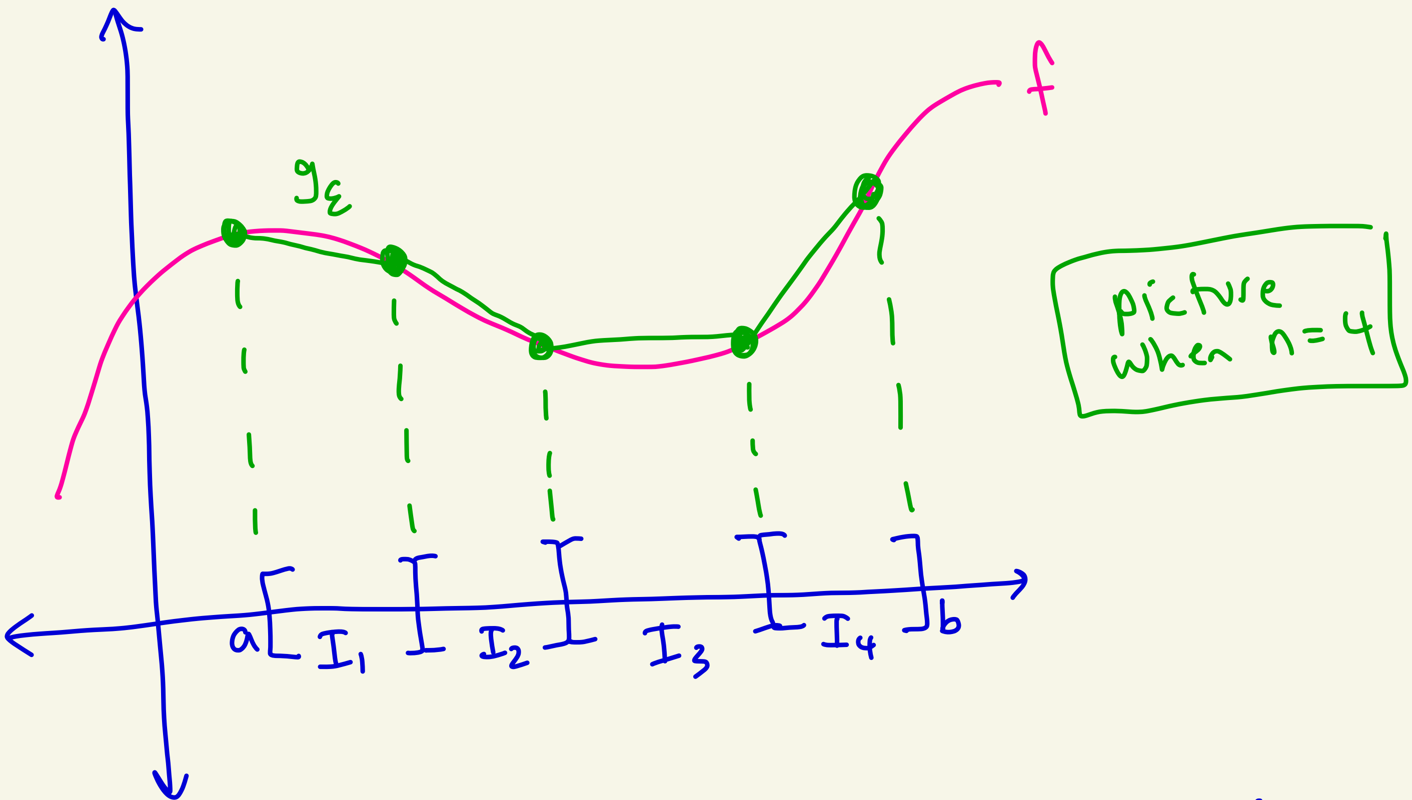
Divide $[a,b]$ into n disjoint intervals of length h, denoted by $I_1, I_2, \ldots, I_n$ where

$$I_1 = [a, a+h]$$
$$I_2 = [a+h, a+2h]$$
$$I_3 = [a+2h, a+3h]$$
$$\vdots$$
$$I_n = [a+(n-1)h, a+nh]$$



On each interval $I_k = (a+(k-1)h, a+kh)$ define g_ε to be the linear function that connects the points

$$(a+(k-1)h, f(a+(k-1)h)) \text{ and } (a+kh, f(a+kh)).$$



By construction, if $x \in I_k$, then $f(x)$ is within $\varepsilon/2$ of $f(a+(k-1)h)$ and $f(a+kh)$.

By the construction of g_ε,

either $|g_\varepsilon(x) - f(a+(k-1)h)| < \varepsilon/2$ (1)

or $|g_\varepsilon(x) - f(a+kh)| < \varepsilon/2$ (2)

If (1) is true, then

$$\begin{aligned} |g_\varepsilon(x) - f(x)| &\le |g_\varepsilon(x) + f(a+(k-1)h)| \\ &\quad + |f(a+(k-1)h) - f(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

If (2) is true, then

$$\begin{aligned} |g_\varepsilon(x) - f(x)| &\le |g_\varepsilon(x) - f(a+kh)| \\ &\quad + |f(a+kh) - f(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

$\blacksquare$